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Quadratics Forms, Rigid Elements, and Formal Power Series Fields

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1. INTRODUCTION

In this paper we look at quadratic forms over fields, and it is always assumed the fields have characteristic different from two. In the study of value sets of binary quadratic forms over fields there are two extreme cases. One is where the form $x^2 - ay^2$ represents all elements in the field F. The set of all such $a \in \dot{F}$ ($\dot{F} = F - \{0\}$) is called the radical and is denoted by $R(F)$. Kaplansky introduced $R(F)$ in [9] and it is also discussed in [1, 3], and [4]. The second case is where $f = x^2 + ay^2$ represents as few elements in $\dot{F}$ as is possible. Clearly $\dot{F}^2 \cup a\dot{F}^2 \subseteq D_F(f)$, where $D_F(f)$ is the set of non-zero elements represented by f over F. If $a \notin R(F)$ and if $x^2 + ay^2$ only represents $\dot{F}^2 \cup a\dot{F}^2$, we say a is a rigid element in $\dot{F}$. Our main result is that if the set consisting of elements which are non-rigid or whose negatives are non-rigid does not comprise all of $\dot{F}$, then F is equivalent with respect to quadratic forms to some formal power series field $K((x))$. Two fields L, K are equivalent with respect to quadratic forms if there exists an isomorphism

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$t: \dot{L}/\dot{L}^2 \to \dot{K}/\dot{K}^2$ such that $t(-1) = -1$ and $t(D_L(\langle a_1, \ldots, a_n \rangle)) = D_K(\langle t(a_1), \ldots, t(a_n) \rangle)$ for all $a_i \in \dot{L}$ and all n. Here $\langle a_1, \ldots, a_n \rangle$ denotes the diagonalized quadratic form $\sum_{i=1}^n a_i x_i^2$ and $t(a)$ refers to any member of the coset $t(a\dot{L}^2)$. It was shown in [2] that $W(L) \cong W(K)$ (where these are the Witt rings of L, K) if and only if L and K are equivalent with respect to quadratic forms.

Most of our notation and terminology will follow what is used in [10].

2. Rigid Elements

First we formally define those elements in $\dot{F}$ which act in an opposite manner to those in $R(F)$.

Definition. Let F be a field. Then x is *rigid* in F if $x \in \dot{F} - R(F)$ and $D_F(\langle 1, x \rangle) = \{1, x\}\dot{F}^2$.

The most common example of a rigid element is the variable x in $K((x))$ providing K is not quadratically closed. In fact it is well known in this case (see [12]) that ax is rigid for all $a \in \dot{K}$. It would be nice if whenever F contained a rigid element, we could say F was equivalent to some $K((x))$. This, as we will see, is not true, but we will get a result almost as good.

Note that a non-trivial radical (i.e., $R(F) \neq \dot{F}^2$) and a rigid element cannot co-exist in F. This follows from the fact that the value set of every form of dimension at least two consists of cosets of $R(F)$ (see [1], [3]). However, this does not mean the definition can be changed to x is rigid if $x \in \dot{F} - \dot{F}^2$ and $D(\langle 1, x \rangle) = \{1, x\}\dot{F}^2$. This weaker criterion allows essentially two more types of fields than the above definition. If the two differ, then there must be an $x \in \dot{F} - \dot{F}^2$ with $D(\langle 1, x \rangle) = \{1, x\}\dot{F}^2$ and $x \in R(F)$. Thus by [3, Proposition 1], $R(F) \subseteq D(\langle 1, x \rangle) = D(\langle 1, 1 \rangle) = \{1, x\}\dot{F}^2 \subseteq R(F)$; and so $R(F) = D(\langle 1, 1 \rangle)$. By induction it follows that $R(F) = \sigma(F) \equiv \bigcup_{n=1}^{\infty} D(n\langle 1 \rangle)$. Consequently, if F is non-real, $R(F) = \dot{F} = \{1, x\}\dot{F}^2$ and $q(F) = 2$. On the other hand, if F is real, then $R(F) = \{1, x\}\dot{F}^2 = \sigma(F) \subsetneq \dot{F}$. Neither of these cases would yield a rigid element under the original definition.

Definition. Let F be a field. Then $x \in \dot{F}$ is *basic* in F if either x or $-x$ is non-rigid. The set $A(F) = \{x \in \dot{F} \mid x \text{ is basic}\}$ is the *basic subset* of F.

Since $D(\langle 1, a \rangle) = D(\langle 1, ar \rangle)$ for $a \in F$, $r \in R(F)$, $A(F)$ consists of cosets of $R(F)$. Also it is clear that $\pm R(F) \subseteq A(F)$. In fact $A(F)$ turns out to be a multiplicative subgroup of $\dot{F}$.

Theorem 1. *For any field F, $A(F)$ is a subgroup of $\dot{F}$.*

Proof. If $A(F) = \dot{F}$, then we are done. Otherwise it suffices to show that if x, y are both non-rigid and if $xy \notin \pm R(F)$, then at least one of xy or $-xy$ is non-rigid. Suppose this is false for some such pair $\{x, y\}$. Clearly we may assume x, $y \notin \pm R(F)$. Then $D(\langle 1, xy\rangle) = \{1, xy\}\dot{F}^2$ and $D(\langle 1, -xy\rangle) = \{1, -xy\}\dot{F}^2$.

Let $B = D(\langle 1, x\rangle) \cap D(\langle 1, g\rangle)$. By [5, Lemma], $B \subseteq D(\langle 1, -xy\rangle) = \{1, -xy\}\dot{F}^2$. But $-xy \notin B$ or else $-xy \in D(\langle 1, y\rangle)$ would imply $-y \in D(\langle 1, xy\rangle) = \{1, xy\}\dot{F}^2$. Contradiction. So $B = \dot{F}^2$.

Now consider the group $D(\langle 1, x, y, xy\rangle)$ of quaternion norms. Since $D(\langle 1, x, y, xy\rangle) = \bigcup D(\langle \alpha, \beta x\rangle)$, where α, $\beta \in D(\langle 1, xy\rangle) = \{1, xy\}\dot{F}^2$, it follows that $D(\langle 1, x, y, xy\rangle) = \{1, y\}\ D(\langle 1, x\rangle) \cup \{1, x\}\ D(\langle 1, y\rangle)$. But no group is the union of two proper subgroups, and so one of $\{1, y\}\ D(\langle 1, x\rangle)$ and $\{1, x\}\ D(\langle 1, y\rangle)$ must contain the other. Due to symmetry, we may assume $\{1, x\}\ D(\langle 1, y\rangle) \subseteq \{1, y\}\ D(\langle 1, x\rangle)$. Because $B = \dot{F}^2$ we must have $D(\langle 1, y\rangle) \subseteq \dot{F}^2 \cup yD(\langle 1, x\rangle)$. Multiplying by y gives $D(\langle 1, y\rangle) \subseteq \{1, y\}\dot{F}^2 \cup D(\langle 1, x\rangle)$. Again $B = \dot{F}^2$ yields $D(\langle 1, y\rangle) \subseteq \{1, y\}\dot{F}^2$. This contradicts y being non-rigid. So we must have either xy or $-xy$ non-rigid, and thus $xy \in A(F)$. ∎

When F is non-real, the situation can be simplified. For instance, by the corollary to Theorem 1 in [5], we see x is rigid if and only if $-x$ is rigid. Hence basic elements are just those in $\dot{F}$ which are not rigid. Moreover, if $A(F) \neq \dot{F}$, then it follows from [6, Corollary 3.6] that $s(F)\langle 1\rangle$ is not universal. Then we can get from Theorem 1 of [5] that every element in $D(s(F)\langle 1\rangle$ is basic. This is obviously true when $A(F) = \dot{F}$.

Remark. If F is non-real, then every member of $D(s(F)\langle 1\rangle)$ is basic.

When F is formally real, basic elements are not necessarily nonrigid. An example of this is given in the last section.

Theorem 1 showed $A(F)$ is a multiplicative group. One might ask whether $\overline{A(F)} = A(F) \cup \{0\}$ forms a field. The answer is generally no, but we can get the following result concerning the additive structure of $\overline{A(F)}$.

PROPOSITION 1. *Let $x, y \in \overline{A(F)}$. If $xy \notin -\dot{F}^2$, then $x + y \in \overline{A(F)}$.*

Proof. Suppose $x + y \notin \overline{A(F)}$. Then clearly $x(x + y) \notin A(F)$, and so $-x(x + y)$ is rigid. Consequently $\langle 1, -x(x + y)\rangle$ represents exactly two cosets of $\dot{F}^2$. But this form clearly represents 1, $-xy$, and $-x(x + y)$; and so two of these must be the same modulo $\dot{F}^2$. But $-xy \not\equiv -x(x + y) \pmod{\dot{F}^2}$ or else $y \equiv x + y \pmod{\dot{F}^2}$ which implies $x + y \in A(F)$. Contradiction. Also $1 \not\equiv -x(x + y) \pmod{\dot{F}^2}$ or else $-x \equiv x + y \pmod{\dot{F}^2}$ yields the same contradiction. Hence $1 \equiv -xy \pmod{\dot{F}^2}$ and the result follows. ∎

The condition $xy \notin -\dot{F}^2$ is obviously not necessary for $x + y \in \overline{A(F)}$ as letting $y = -x$ shows. A less trivial example is given by $F = R((x))$, where

$A(F) = \pm\dot{F}^2$. Then both $-1+x^2$ and $1-x^3$ lie in $A(F)$, their product is in $-\dot{F}^2$, and their sum is also contained in $A(F)$.

Suppose $x \in A(F) - (-\dot{F}^2)$ and $z \in \dot{F} - A(F)$. If $z \in D(\langle 1, x\rangle)$, then $-x \in D(\langle 1, -z\rangle)$. But $z \notin A(F)$ means both z, $-z$ are rigid. Hence $-x \in \{1, -z\}\dot{F}^2$ and the restrictions on x imply $x \in z\dot{F}^2$. This contradicts $z \notin A(F)$. We conclude then that $D(\langle 1, x\rangle) \subseteq A(F)$. From this it is easy to see $D(\langle x, y\rangle) \subseteq A(F)$ for $x, y \in A(F)$ and $\langle x, y\rangle$ anisotropic. Using the fact that $D(\varphi \perp \langle x\rangle) = \bigcup_{a \in D(\varphi)} D(\langle a, x\rangle)$, Theorem 1, and induction, it now follows that $D(\langle x_1,..., x_n\rangle) \subseteq A(F)$ for all $x_i \in A(F)$, $1 \leqslant i \leqslant n$, and $\langle x_1,..., x_n\rangle$ anisotropic. In fact the above arguments also hold if $A(F)$ is replaced by any multiplicative subgroup containing $A(F)$.

PROPOSITION 2. *If $\varphi = \langle x_1,..., x_n\rangle$ is anisotropic and $x_i \in A(F)$, $1 \leqslant i \leqslant n$, then $D(\varphi) \subseteq A(F)$.*

DEFINITION. *If F is a field, let $T(F)$ denote the canonical image in $W(F)$ of the set of all (isometry classes of) forms φ with $D(\varphi_{\text{an}}) \subseteq A(F)$, where φ_{an} is the anisotropic part of φ.*

If φ is anisotropic and $D(\varphi) \subseteq A(F)$, then any diagonalization of φ has all its coefficients lying in $A(F)$. By Proposition 2 if $\Psi = \langle x_1,..., x_n\rangle$ is anisotropic and $x_i \in A(F)$, $1 \leqslant i \leqslant n$, then $D(\Psi) \subseteq A(F)$. Thus $T(F)$ can be thought of as the image in $W(F)$ of $\langle 1, -1\rangle$ and all anisotropic diagonalized forms with coefficients in $A(F)$. The next result, which shows even more is true, will enable us to conclude that $T(F)$ is a subring of $W(F)$.

PROPOSITION 3. *Let $\varphi = \langle x_1,..., x_n\rangle$ and $\varphi_{\text{an}} = \langle y_1,..., y_m\rangle$. If $x_i \in A(F)$ for all i, then $y_j \in A(F)$ for all j.*

Proof. We proceed by induction on n. The case $n = 1$ is clear. Suppose $n > 1$ and let k, $1 \leqslant k \leqslant n$, be the biggest integer such that $\Psi = \langle x_1,..., x_k\rangle$ is anisotropic. If $k = n$, we are done by Proposition 2. If $k < n$, then $-x_{k+1} \in D(\Psi)$; and again by Proposition 2, there exist $z_1,..., z_{k-1} \in A(F)$ such that $\Psi \cong \langle z_1,..., z_{k-1}, -x_{k+1}\rangle$. Thus $\varphi \cong \varphi' \perp \langle x_{k+1}, -x_{k+1}\rangle$, where $\varphi' = \langle z_1,..., z_{k-1}, x_{k+2},..., x_n\rangle$. Then $\varphi_{\text{an}} \cong \varphi'_{\text{an}}$ and the induction hypothesis gives the result. ∎

Proposition 3 allows us to consider $T(F)$ as the image in $W(F)$ of all diagonalized forms whose coefficients lie in $A(F)$. Clearly then $T(F)$ is closed under addition. Also since $A(F)$ is a multiplicative subgroup, the next result is readily apparent.

PROPOSITION 4. *$T(F)$ is a subring of $W(F)$.*

In the terminology of [8], a subring $R \subseteq W(F)$ is excellent if any anisotropic form φ whose image is in R can be expressed as an orthogonal sum of 1-dimensional forms in R. So $T(F)$ is an excellent subring of $W(F)$.

We would like to be able to discuss the structure of $W(F)$ in terms of $T(F)$. The next proposition is the key that will allow us to do so. Note the similarity between it and what holds for formal power series fields.

PROPOSITION 5. *If $f_1,\dots,f_n$ are diagonalized forms with coefficients in $A(F)$ and if $y_1,\dots,y_n \in \dot{F}$ are pairwise distinct modulo $A(F)$, then*

$$D(y_1f_1 \perp \cdots \perp y_nf_n) = \bigcup_{i=1}^{n} y_iD(f_i).$$

Proof. This is clear for $n=1$. Now suppose $a, b \in A(F)$ and $z \notin A(F)$. Then abz is rigid and $D(\langle a, bz\rangle) = \{a, bz\}\dot{F}^2$. Applying this to $n=2$, we obtain $D(y_1f_1 \perp y_2f_2) = \bigcup_{\alpha_i \in D(f_i)} D(\langle y_1\alpha_1\rangle \perp \langle y_2\sigma_2\rangle) = \bigcup_{\alpha_i \in D(f_i)} \{y_1\alpha_1, y_2\alpha_2\}\dot{F}^2 = y_1D(f_1) \cup y_2D(f_2)$. The general inductive step is similar and follows straightforwardly by applying the $n=2$ case. ∎

COROLLARY. *With the hypothesis as in Proposition 5, $y_1f_1 \perp \cdots \perp y_nf_n$ is isotropic if and only if at least one f_i is isotropic.*

Proof. $y_1f_1 \perp \cdots \perp y_nf_n$ is isotropic if and only if $-D(y_1f_1) \cap D(y_2f_2 \perp \cdots \perp y_nf_n) \neq \varnothing$. But since $-1 \in A(F)$, Proposition 5 shows this intersection is empty if all the f_i are anisotropic. ∎

For any set $S \subseteq \dot{F}$, let $\langle S\rangle$ denote the subgroup of $\dot{F}/\dot{F}^2$ generated by $\{a\dot{F}^2 \mid a \in S\}$. Let B be a subset of $\dot{F}$ which is independent modulo $A(F)$ and such that $\dot{F}/\dot{F}^2$ is generated by $A(F)/\dot{F}^2$ and $\langle B\rangle$. That is, $\dot{F}$ is generated independently by $A(F)$ and the elements in B. Note that $\langle B\rangle \cong \dot{F}/A(F)$.

PROPOSITION 6. *If $\{y_i\}_{i\in I}$ is a set of representatives in F of the elements in $\langle B\rangle$, then $W(F)$ as a group is given by $\oplus_{i\in I} y_iT(F)$.*

Proof. Since every element of $\dot{F}$ is a product of a y_i and some element in $A(F)$, $W(F)$ is a sum of the subgroups $y_iT(F)$, $i \in I$. To show this sum is direct, it suffices to show $y_1f_1 \perp \cdots \perp y_nf_n$ is not hyperbolic if the y_i are distinct and each f_i is a diagonalized, anisotropic form with coefficients in $A(F)$. But this follows immediately from the last corollary. ∎

We are now in a position to determine the ring structure of $W(F)$ in terms of $T(F)$. We will need the notion of a group ring. If R is a commutative ring and G is a group, then the group ring of R over G is denoted by $R[G]$.

THEOREM 2. $W(F) \cong T(F)\ [\langle B\rangle] \cong T(F)\ [\dot{F}/A(F)]$.

Proof. Let $\{y_i\}_{i\in I}$ be as in Proposition 6. Then every element in $T(F)$ $|\langle B\rangle|$ can be written as $\beta = \sum_{j=1}^{n} \bar{f}_j(y_{i_j}\dot{F}^2)$, where each f_j is a diagonalized, anisotropic form with coefficients in $A(F)$ and $\bar{f}_j$ is its image in $W(F)$. Define $\sigma\colon T(F)\ [\langle B\rangle] \to W(F)$ by $\sigma(\beta) = \sum_{j=1}^{n} \overline{(y_{i_j} f_j)}$. It is easy to show σ is a ring homomorphism. By Proposition 6, σ is onto; and by the corollary to Proposition 5, σ is $1-1$. ∎

Note that if $A(F) \neq \dot{F}$, then $\langle B\rangle$ is a group of exponent 2. Thus we can write $\langle B\rangle = G \oplus H$, where H is a subgroup of order 2, and hence $W(F) \cong T(F)\ [G \oplus H] \cong (T(F)[G])\ [H]$. This is similar to the case $F = K((x))$, where $W(F) = W(K)\ [\{\langle \bar{1}\rangle, \langle \bar{x}\rangle\}]$. In the next section we will make the connection between these situations precise.

Recall that the generalized u-invariant, $u(F)$, of Elman and Lam [6] is the maximum dimension of all anisotropic forms whose images in $W(F)$ are torsion. If no such maximum exists, $u(F) = \infty$. We can also define the same concept over $A(F)$. That is, $u(A)$ is the maximum dimension of all diagonalized, anisotropic forms with coefficients in $A(F)$ which are torsion.

THEOREM 3. *If F is not a real, Pythogorean field, then $u(F) = |\dot{F}/A(F)|\ u(A)$.*

Proof. If $u(A) = \infty$, then clearly $u(F) = \infty$. We claim $u(A) \geqslant 1$. This is obvious if F is non-formally real. If F is real, then since F is not Pythagorean, there is a $y \in D(\langle 1, 1\rangle) - \dot{F}^2$. Hence $-1 \in D(\langle 1, -y\rangle)$ and $-1 \notin \{1, -y\}\dot{F}^2$. Consequently $-y$ is not rigid and so $-y \in A(F)$. But clearly $\langle 1, -y\rangle$ is torsion in $T(F)$. Thus $u(A) \geqslant 2$. It now follows immediately form Proposition 5's corollary that $u(F) = \infty$ if $|\dot{F}/A(F)| = \infty$.

We can now assume $1 \leqslant u(A)$, $|\dot{F}/A(F)| < \infty$. Again from Proposition 5's corollary, we have $u(F) \geqslant |\dot{F}/A(F)|\ u(A)$. Suppose φ is any anisotropic torsion form over F. Then by Proposition 6, there are diagonalized, anisotropic forms f_i with coefficients in $A(F)$ and there are $y_i \in \dot{F}$ such that $\bar{\varphi} = \overline{y_1 f_1} + \cdots + \overline{y_n f_n}$. If $\bar{\varphi}$ has order m, then $\overline{my_1 f_1} + \cdots + \overline{my_n f_n} = 0$ and the directness of the sum in Proposition 6 shows $\overline{my_i f_i} = 0$ for all i. Thus $\dim f_i \leqslant u(A)$ and $\dim \varphi \leqslant |\dot{F}/A(F)|\ u(A)$. Since φ was any anisotropic, torsion form over F, $u(F) \leqslant |\dot{F}/A(F)|\ u(A)$. ∎

When F is a real, Pythagorean field, then $u(F) = u(A) = 0$. So the conclusion in Theorem 3 holds here too providing $|\dot{F}/A(F)| < \infty$. All parts of the following corollary are direct consequences of the above remark and Theorem.

COROLLARY. *Suppose $u(F) < \infty$ and F is not a real, Pythagorean field. Then*

(i) *$|\dot{F}/A(F)|$ divides $u(F)$.*

(ii) *If* $A(F) \neq \dot{F}$, *then* $u(F)$ *is even.*

(iii) $|\dot{F}/A(F)| \leqslant u(F)/2$ *if and only if* $A(F) \neq \dot{F}^2$ *(this holds if* $s \geqslant 2$*).*

By (ii) we see that if $u(F) < \infty$ for a non-formally real field and if F contains a rigid element, then $u(F)$ is even. This is obviously still a long ways from Kaplansky's conjecture about $u(F)$ always being a 2-power.

3. $W(F)$ AND GROUP RINGS

In [13] characterizations were given for fields whose Witt rings were isomorphic to group rings of the form $Z/nZ[G]$. Here we consider the case when $W(F) = R[G]$, where R is a subring (containing $\langle 1 \rangle$) of $W(F)$ and G is a subgroup of order 2. The next proposition shows we may always assume G has a very simple form.

In order to simplify notation, we will from now on identify forms over F and their images in $W(F)$. Which is meant will always be clear by the context.

PROPOSITION 7. *Suppose* $W(F) = R[G]$, *where* $|G| = 2$. *Then there is an* $a \in \dot{F}$ *such that* $W(F) = R[\{\langle 1 \rangle, \langle a \rangle\}]$.

Proof. Assume $G = \{\langle 1 \rangle, q\}$ where $q = \langle a_1, ..., a_n \rangle$. For each i, $1 \leqslant i \leqslant n$, there are r_i, $s_i \in R$ so that $\langle a_i \rangle = r_i + s_i q$. Thus $q = (\sum_{i=1}^n r_i) + (\sum_{i=1}^n s_i)q$ and $\sum_{i=1}^n s_i = \langle 1 \rangle$. Hence some s_i, say s_1, has odd dimension. In particular s_1 is not a zero-divisor (see [10, p. 250]). Also $q^2 = \langle 1 \rangle$ and $\langle a_1 \rangle = r_1 + s_1 q$ imply that dim r_1 is even. Moreover, $\langle 1 \rangle = \langle a_1 \rangle^2 = (r_1^2 + s_1^2) + (2r_1 s_1)q$ shows that $2r_1 = 0$ because s_1 is not a zero-divisor. We have then that r_1 is an even-dimensional, torsion form and hence must be nilpotent (see [10, p. 248]). From the above, we see $s_1^2 = \langle 1 \rangle - r_1^2$ and so s_1^2 (and therefore s_1) is a unit in R. Choose $p_1 \in R$ so that $p_1 s_1 = \langle 1 \rangle$. Then $q = -p_1 r_1 + p_1 \langle a_1 \rangle$, and so $\{\langle 1 \rangle, \langle a_1 \rangle\}$ generates $W(F)$ over R. To show $W(F) = R[\{\langle 1 \rangle, \langle a_1 \rangle\}]$, all that remains is to prove $\langle 1 \rangle$ and $\langle a_1 \rangle$ are independent over R. If r, $s \in R$ with $r + s\langle a_1 \rangle = 0$, then $r + s(r_1 + s_1 q) = 0$. Consequently, $ss_1 = r + sr_1 = 0$ and s_1 not being a zero-divisor gives $r = s = 0$. ∎

For the next five lemmas, we assume $W(F) = R[G]$, where $G = \{\langle 1 \rangle, \langle a \rangle\}$. Also for the remainder of this section, let $K = F(a^{1/2})$; and denote the canonical map of $W(F)$ into $W(K)$ by i. It is well known that $\ker i = \langle 1, -a \rangle W(F)$. Finally $i(\dot{F}/\dot{F}^2)$, we mean $i\{\langle b \rangle | b \in \dot{F}\}$.

LEMMA 1. *If* $x \in \dot{F}$ *with* $\langle x \rangle \in R$, *then* $D(\langle 1, ax \rangle) = \{1, ax\}\dot{F}^2$. *In particular* $D(\langle 1, a \rangle) = \{1, a\}\dot{F}^2$ *and* $D(\langle 1, -a \rangle) = \{1, -a\}\dot{F}^2$.

Proof. Since $\langle x \rangle \in R$, $W(F) = R[\{\langle 1 \rangle, \langle ax \rangle\}]$. Set $b = ax$ and suppose

$c \in D(\langle 1, b\rangle)$. Then $\langle 1, b\rangle \cong \langle c, bc\rangle$. There are unique $r_i \in R$ such that $\langle c\rangle = r_0 + r_1\langle b\rangle$, and hence $\langle 1\rangle + \langle b\rangle = (r_0 + r_1)(\langle 1\rangle + \langle b\rangle)$. Thus $r_0 + r_1 = \langle 1\rangle$ and $\langle c\rangle = r_0 + (1 - r_0)\langle b\rangle$. Taking determinants shows $\langle c\rangle = \langle \pm b^n\rangle$ with $n = 0$ or 1. If $\langle c\rangle = \langle -1\rangle$, then $\langle 1, b\rangle \cong \langle -1, -b\rangle$ implies $2\langle 1, b\rangle = 0$ in $W(F) = R[\{\langle 1\rangle, \langle b\rangle\}]$. Therefore $2 = 0$ in $W(F)$ and $\langle -1\rangle = \langle 1\rangle$. Similarly if $\langle c\rangle = \langle -b\rangle$, then $\langle -b\rangle = \langle b\rangle$. So $c \in \{1, b\}\dot{F}^2$. ∎

LEMMA 2. $R \cong i(W(F))$.

Proof. It is known that $i(W(F)) \cong W(F)/\langle 1, -a\rangle\, W(F)$. Let σ: $R[\{\langle 1\rangle, \langle a\rangle\}] \to R$ be the ring homomorphism defined by $\sigma(r_0 + r_1\langle a\rangle) = r_0 + r_1$. Then σ is a surjection with $\ker \sigma = (\langle 1\rangle - \langle a\rangle)\ R[G] = \langle 1, -a\rangle$ $W(F)$. Thus $R \cong W(F)/\langle 1, -a\rangle\, W(F)$. ∎

LEMMA 3. $\operatorname{Ann}(\langle 1, a\rangle) = \langle 1, -a\rangle\, W(F) = \langle 1, -a\rangle R$.

Proof. If $g \in \operatorname{Ann}(\langle 1, a\rangle)$, then $g + g\langle a\rangle = 0$ in $W(F)$. Write $g = r_0 + r_1\langle a\rangle$ with $r_i \in R$. Then $(r_0 + r_1)(\langle 1\rangle + \langle a\rangle) = 0$, and hence $r_0 + r_1 = 0$. Thus $g = r_0 - r_0\langle a\rangle = r_0\langle 1, -a\rangle$. ∎

LEMMA 4. *For any* $b \in \dot{K}$ *with* $\langle b\rangle \notin i(\dot{F}/\dot{F}^2)$, $W(K) = i(W(F))$ $[\{\langle 1\rangle, \langle b\rangle\}]$.

Proof. Since $D_F(\langle 1, -a\rangle) = \{1, -a\}\dot{F}^2$, we have the following exact sequence:

$$1 \to \{\dot{F}^2, a\dot{F}^2\} \to \dot{F}/\dot{F}^2 \xrightarrow{i} \dot{K}/\dot{K}^2 \xrightarrow{N_{K/F}} \{\dot{F}^2, -a\dot{F}^2\} \to 1.$$

Thus $i(\dot{F}/\dot{F}^2)$ has index 2 in $\dot{K}/\dot{K}^2$, and so every $c \in \dot{K}$ satisfies $\langle c\rangle = \langle d\rangle\langle b^n\rangle$, where $\langle d\rangle \in i(\dot{F}/\dot{F}^2)$ and $n = 0$ or 1. Hence $W(K)$ is generated as an $i(W(F))$-module by $\langle 1\rangle$ and $\langle b\rangle$. It remains to show $\langle 1\rangle$, $\langle b\rangle$ are linearly independent over $i(W(F))$. Suppose $g_0 + g_1\langle b\rangle = 0$ with $g_j \in i(W(F))$. It follows then that $\langle b\rangle(g_0 - g_1) = g_0 - g_1$. Choose φ over F so that $i(\varphi) = g_0 - g_1$. Then $b\varphi \cong \varphi$ over K. Since $\langle b\rangle \notin i(\dot{F}/\dot{F}^2)$, the exact sequence above yields $N_{K/F}(b) \in -a\dot{F}^2$. So by Scharlau's norm principle, $-a\varphi \cong \varphi$ over F. Hence $\varphi \in \operatorname{Ann}(\langle 1, a\rangle) = \langle 1, -a\rangle$ $W(F)$. Therefore $i(\varphi) = 0$ which means $g_0 = g_1$ and $-bg_0 \cong g_0$ over K. Using the exact same argument on g_0 as was used above on $g_0 - g_1$, we find $g_0 = 0$. ∎

A 2-extension of any field F is a field extension of F lying in a quadratic closure of F.

LEMMA 5. *Let* L *be a finite 2-extension of* K *such that* $i(\dot{F}/\dot{F}^2)$ *injects into* $\dot{L}/\dot{L}^2$ *under the natural map* ε: $\dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$. *Then if* $\langle b\rangle \in \dot{L}/\dot{L}^2 - \varepsilon(i(\dot{F}/\dot{F}^2))$, $W(L) = \varepsilon(iW(F))$ $[\{\langle 1\rangle, \langle b\rangle\}]$.

Proof. We proceed by induction on $[L:K]$. If $[L:K]=1$, the result comes from Lemma 4. So suppose $[L:K]>1$ and write $L=K_1(c^{1/2})$ with $K\subseteq K_1$ and $c\notin \dot{K}_1^2$. Let $\varepsilon_1:\dot{K}/\dot{K}^2\to\dot{K}_1/\dot{K}_1^2$ and $\varepsilon_2:\dot{K}_1/\dot{K}_1^2\to\dot{L}/\dot{L}^2$ be the natural maps (so $\varepsilon=\varepsilon_2\circ\varepsilon_1$). Since $i(\dot{F}/\dot{F}^2)$ injects into $\dot{L}/\dot{L}^2$, it follows that $\langle c\rangle\notin\varepsilon_1(i(\dot{F}/\dot{F}^2))$. By the induction hypothesis then, $W(K_1)=\varepsilon_1(i(W(F)))$ $[\{\langle 1\rangle,\langle c\rangle\}]$; and from this we see $\varepsilon_1(i(\dot{F}/\dot{F}^2))$ has index 2 in $\dot{K}_1/\dot{K}_1^2$. Since ε_2 must be an injection on $\varepsilon_1(i(\dot{F}/\dot{F}^2))$ but is not an injection on $\dot{K}_1/\dot{K}_1^2$, we must now have $\varepsilon(i(\dot{F}/\dot{F}^2))=\varepsilon_2(\dot{K}_1/\dot{K}_1^2)$. Hence $\varepsilon(i(W(F))=\varepsilon_2(W(K_1))$. So $\langle b\rangle\notin\varepsilon_2(\dot{K}_1/\dot{K}_1^2)$ and by Lemma 4, $W(L)=\varepsilon_2(W(K_1))$ $[\{\langle 1\rangle,\langle b\rangle\}]$. ∎

We are ready now for a characterization of fields satisfying $W(F)\cong R[G]$. They turn out to be equivalent to formal power series fields.

THEOREM 4. *If $W(F)=R[G]$, where G is of order 2, then there is a field k such that F is equivalent to $k((t))$ with respect to quadratic forms.*

Proof. By Proposition 7, we may assume $G=\{\langle 1\rangle,\langle a\rangle\}$. We must show $W(F)\cong W(k((t)))$ for some field k. Consider the set of 2-extensions of K in a given quadratic closure of K given by $T=\{k\,|\,i(\dot{F}/\dot{F}^2)$ injects into $\dot{k}/\dot{k}^2\}$. Clearly $T\neq\varnothing$ since $K\in T$. Partially order T by set inclusion and suppose $\{K_i\}_{i\in I}$ is a chain. Let $L=\bigcup K_i$. Since any two elements in L lie in at least one K_i, $i(\dot{F}/\dot{F}^2)$ injects into $\dot{L}/\dot{L}^2$. Thus $L\in T$. By Zorn's lemma, T has a maximal element k.

Claim. $i(\dot{F}/\dot{F}^2)\cong\dot{k}/\dot{k}^2$ under the natural map $\varepsilon:\dot{K}/\dot{K}^2\to\dot{k}/\dot{k}^2$.

By $k\in T$, ε restricted to $i(\dot{F}/\dot{F}^2)$ is $1-1$. Let $c\in\dot{k}-\dot{k}^2$. Then $k_1=k(c^{1/2})$ is a proper 2-extension of k, and by the maximality of k, $i(\dot{F}/\dot{F}^2)$ does not inject into $\dot{k}_1/\dot{k}_1^2$. So there exists $\langle x\rangle\in\dot{F}/\dot{F}^2$ such that $\langle x\rangle\neq\langle 1\rangle$ in $\dot{k}/\dot{k}^2$ but $\langle x\rangle=\langle 1\rangle$ in $\dot{k}_1/\dot{k}_1^2$. The only way this can happen is if $\langle x\rangle=\langle c\rangle$ in $\dot{k}/\dot{k}^2$. Thus ε is a surjection, and the claim is established.

To prove the theorem, it suffices, in view of Lemma 2, to show that $i(W(F))\cong W(k)$. Extending ε to a map from $i(W(F))$ to $W(k)$, we see from the claim that ε is surjective. Suppose $q\in i(W(F))$ satisfies $\varepsilon(q)=0$. Then there is a finite 2-extension L of K with $L\subseteq k$ and $q=0$ in $W(L)$. Choose such an L with $[L:K]$ minimal. If $L\neq K$, there is a K_1 with $K\subseteq K_1\subsetneq L$ and $L=K_1(b^{1/2})$. Then $\langle b\rangle\notin\varepsilon_1(i(\dot{F}/\dot{F}^2))$, where $\varepsilon_1:\dot{K}/\dot{K}^2\to\dot{K}_1/\dot{K}_1^2$, and so by Lemma 5, $W(K_1)=\varepsilon_1(i(W(F)))$ $[\{\langle 1\rangle,\langle b\rangle\}]$. Now q is hyperbolic over L and so by [11, 2.2.9], there is a form q' over K_1 so that $q=\langle 1,-b\rangle q'$ in $W(K_1)$. But then $q+q\langle b\rangle=0$ in $W(K_1)$ and $q\in\varepsilon_1(i(W(F)))$ yield $q=0$ in $W(K_1)$. This contradicts the minimality of $[L:K]$. Thus we must have $L=K$ and $q=0$ in $W(K)$. So ε is $1-1$. ∎

4. The Main Theorem

The following main result is a consequence of Theorem 3, the discussion immediately following Theorem 2, and Theorem 4.

THEOREM 5. *If $A(F) \neq \dot{F}$, then there is a field k such that F and $k((t))$ are equivalent with respect to quadratic forms.*

In light of some previous remarks, Theorem 5 shows that any non-real field containing a rigid element is equivalent to some $k((t))$. However, this is not true for real fields. For let F be a Pythagorean field with exactly 3 orderings and $q(F)=8$ (see [7, p. 1187]). Let $x \in \dot{F}$ be positive at two orderings and negative at the third. If $y \in D(\langle 1, x\rangle)$, then clearly y is positive at the orderings x is. Thus $y \in \{1, x\}\dot{F}^2$ by [7, Proposition 4.1] and x is rigid. But F cannot be equivalent to a $k((t))$ since all power series fields have an even number of orderings. This is also an example of a field in which x is simultaneously rigid and basic.

By using Theorem 5 and induction, it is easy to see that when $|\dot{F}/A(F)| = 2^n < \infty$, there is a field k such that F is equivalent to $k((x_1)) \cdots ((x_n))$ and $A(k)=\dot{k}$. Actually by employing techniques from [1], one can show there is a 2-extension k of F such that $A(k)=\dot{k}$ and $W(F) \cong W(k)$ $[\dot{F}/A(F)]$ regardless of the cardinality of $\dot{F}/A(F)$.

The notion of rigid element can be generalized. We call $x \in F$ semirigid if $x \in \dot{F} - R(F)$ and $D(\langle 1, x\rangle) = \{1, x\}R(F)$. This allows the concept of rigidity with a non-trivial radical. All the results, with the exception of Proposition 1, through Proposition 4 still hold where now $A(F)$ is the set of all elements in $\dot{F}$ such that at least one of x or $-x$ is not semi-rigid. Even Proposition 1 holds if $-\dot{F}^2$ is replaced by $-R(F)$. Proposition 5, however, does not hold as a 1-dimensional f_i wrecks any attempt to fix up the conclusion.

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